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Linear Differential Equations of Infinite Order and Theta Functions

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0. INTRODUCTION

The purpose of this article is to show that some finiteness theorem (= finite dimensionality of the space of solutions) holds for a class of systems of linear differential equations of *infinite order*. Although finiteness theorems for holonomic systems of (micro-)differential equations of finite order have recently become quite popular, the character of the theorems which we present here is different from the results for equations of finite order. Hence, in this introduction, we discuss a simple and instructive example so that it may help the reader's understanding of the character of the results in this article. As the example will indicate, our results have close connection with the celebrated result of Hamburger on the characterization of the ζ-function of Riemann, although we deal with theta functions (Hamburger [2], Hecke [3], and Weil [8]; see also Ehrenpreis and Kawai [1]). This connection was pointed out to one of us (T.K.) by Professor L. Ehrenpreis. Concerning the basic properties of linear differential operators of infinite order, we refer the reader to Sato–Kawai–Kashiwara [6, Chap. II][1] (hereafter referred to as S–K–K). Here we only emphasize that a linear differential operator of infinite order acts upon the sheaf of holomorphic functions as a sheaf homomorphism. Hence our main result (Theorem 2.14 in Section 2) is of local character. This forms a striking contrast to the hitherto known way of characterizing theta functions through their automorphic properties.

Now, in order to provide an example of our results, let us show how the theta zero-value (Nullwerte) is related to a system of linear differential equations of infinite order. In order to fix the notations, let us consider

$$h(\tau)=\sum_{\nu\in Z}\exp(\pi\sqrt{-1}\,\nu^2\tau) \qquad (0.1)$$

[1] Note, however, that in accordance with the notations used in recent literature, we use $\mathscr{D}_X^\infty$ (resp., $\mathscr{D}_X$) to denote the sheaf of linear differential operators of infinite (resp., finite) order. The quoted article uses $\mathscr{D}_X$ (resp., $\mathscr{D}_X^f$) instead of $\mathscr{D}_X^\infty$ (resp., $\mathscr{D}_X$). Note also that all operators considered here are with holomorphic coefficients.

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on the domain $\mathbb{C}^+ =_{\text{def}} \{\tau \in \mathbb{C}; \operatorname{Im} \tau > 0\}$. Then, at least formally, $h(\tau)$ is annihilated by the following infinite product of linear differential operators:

$$Q_1 = \frac{d}{d\tau} \prod_{\nu=1}^{\infty} \left(1 - \frac{1}{\pi\sqrt{-1}\,\nu^2} \frac{d}{d\tau}\right). \tag{0.2}$$

Since we know

$$\sinh \zeta = \zeta \prod_{\nu=1}^{\infty} \left(1 + \frac{\zeta^2}{\pi^2 \nu^2}\right), \tag{0.3}$$

again, formally, we find

$$\begin{aligned} Q_1 &= \sqrt{\pi\sqrt{-1}\,\frac{d}{d\tau}} \sinh \sqrt{\pi\sqrt{-1}\,\frac{d}{d\tau}} \\ &= \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} \left(\pi\sqrt{-1}\,\frac{d}{d\tau}\right)^{j+1}. \end{aligned} \tag{0.4}$$

Although the above reasoning is a heuristic one, the resulting operator Q_1 (understood as the right hand side of (0.4)) is a well-defined linear differential operator of infinite order, and

$$Q_1\left(\frac{d}{d\tau}\right) h(\tau) = 0 \tag{0.5}$$

holds on $\mathbb{C}^+$. However, this equation only cannot characterize $h(\tau)$, because any function of the form

$$\sum_{\nu \in \mathbb{Z}} a_\nu \exp(\pi\sqrt{-1}\,\nu^2 \tau) \tag{0.6}$$

satisfies Eq. (0.5) if it converges absolutely and uniformly on each compact subset of $\mathbb{C}^+$. Needless to say, this infinite dimensionality of the solutions of Eq. (0.5) is due to the fact that the operator Q_1 is of infinite order.

In passing, Jacobi's imaginary transformation tells us

$$h(\tau) = \exp(\pi\sqrt{-1}/4)\, \tau^{-1/2} h(-1/\tau). \tag{0.7}$$

By applying the same reasoning as above to the right hand side of (0.7), we obtain another equation,

$$Q_2\left(\tau, \frac{d}{d\tau}\right) h(\tau) = 0, \tag{0.8}$$

where

$$Q_2 = \left(\tau \frac{d}{d\tau} + \frac{1}{2}\right) \frac{\sinh \sqrt{\pi \sqrt{-1}(\tau^2(d/d\tau) + \tau/2)}}{\sqrt{\pi \sqrt{-1}(\tau^2(d/d\tau) + \tau/2)}}.$$

Again, Q_2 only cannot characterize $h(\tau)$, because any function of the form

$$\sum_{\nu} b_\nu \tau^{-1/2} \exp(-\pi \sqrt{-1}\, \nu^2/\tau) \tag{0.9}$$

satisfies (0.8) if it converges absolutely and uniformly. However, if we consider Eqs. (0.5) and (0.8) simultaneously, we may expect some finiteness theorem for the equations. Showing that it is really the case is the aim of this article (Theorem 2.14 in Section 2). Although we have so far considered equations with one unknown function, using equations with several unknown functions is more advantageous in developing the general theory. For example, if we introduce $\mathbf{h}(\tau) = {}^t(h_1(\tau), h_2(\tau))$, where $h_1(\tau) = h(\tau)$ and $h_2(\tau) = \sum_\nu 2\pi \sqrt{-1}\, \nu \exp(\pi \sqrt{-1}\, \nu^2 \tau)$ $(=0)$, then the equations corresponding to (0.5) and (0.8) take the following form,

$$\left(\exp P_j \left(\tau, \frac{\partial}{\partial \tau}\right)\right) \mathbf{h}(\tau) = \mathbf{h}(\tau) \qquad (j = 1, 2), \tag{0.10}$$

where

$$P_1 \left(\tau, \frac{\partial}{\partial \tau}\right) = \begin{pmatrix} & 1 \\ 4\pi \sqrt{-1} \dfrac{\partial}{\partial \tau} & \end{pmatrix} \tag{0.11}$$

and

$$P_2 \left(\tau, \frac{\partial}{\partial \tau}\right) = \begin{pmatrix} & \tau \\ 4\pi \sqrt{-1} \left(\tau \dfrac{\partial}{\partial \tau} + \dfrac{1}{2}\right) & \end{pmatrix}. \tag{0.12}$$

Since Eq. (0.10) is more symmetric than (0.5) and (0.8), we formulate our results using the matrix notations. We end this introduction by noting the following properties (A) and (B) of P_j's. A suitable generalization of these properties is the starting point of the reasoning in Section 2.

(A) Every component of $(aP_1 + bP_2)^2$ $(a, b \in \mathbb{C})$ is a linear differential operator of order at most one; that is, $\operatorname{ord}(aP_1 + bP_2)$ is (at most) 1/2 in the sense of Definition 1.1(ii) of Section 1.

This guarantees, in particular, that each component of $\exp P_j$ $(j = 1, 2)$ is a linear differential operator of infinite order.

(B) $[P_1, P_2] = 2\pi\sqrt{-1}\, I_2$ holds. Here I_2 denotes the 2×2 identity matrix.

This guarantees

$$(\exp P_1 - 1)(\exp P_2 - 1) = (\exp P_2 - 1)(\exp P_1 - 1). \tag{0.13}$$

(See Theorem 1.4 in Section 1.) Hence system (0.10) is in involution.

The essential part of this article was announced in Sato [5] with more emphasis on the microlocal aspect of the problem.

List of notations

X:	A complex manifold.
$\mathscr{D}_X$:	The sheaf of linear differential operators of finite order on X. The subscript X is often omitted in this symbol and also in other symbols given below.
$\mathscr{D}_X^\infty$:	The sheaf of linear differential operators of infinite order on X.
$\mathscr{D}_X(m)$:	The sheaf of linear differential operators of order equal to or at most m on X.
$M_r(\mathscr{D}_X)$, $M_r(\mathscr{D}_X^\infty)$:	The sheaf of $r \times r$ matrices whose components belong to $\mathscr{D}_X$ or $\mathscr{D}_X^\infty$.
ord P for P in $M_r(\mathscr{D}_X)$:	See Definition 1.1(ii), in Section 1.

1. A Composition Rule for $\exp P$'s

The purpose of this section is to prove a variant of Campbell–Hausdorff formula in its simplest form (Theorem 1.4.) The results in this section will be used in Section 2 in an essential manner. As we will see by examples given in Section 3, it is inevitable to formulate the problem modulo some $\mathscr{D}$-module.

Let us first prepare some notations. In what follows, X denotes a complex manifold.

Definition 1.1. Let P be an $r \times r$ matrix of linear differential operators on X. Let $P_{i,j}$ $(1 \leqslant i, j \leqslant r)$ denote its (i,j) component.

(i) comp-ord P is, by definition, $\max_{1 \leqslant i,j \leqslant r} \operatorname{ord} P_{i,j}$. Here $\operatorname{ord} P_{i,j}$ denotes the order of the differential operator $P_{i,j}$.

(ii) If there exist real numbers α and c such that

$$\text{comp-ord}\, P^k \leqslant [\alpha k] + c \tag{1.1}$$

holds for every non-negative integer k, then we say that the order of the matrix P is α, and we denote it by $\operatorname{ord} P$. Here P^k denotes the kth power of P and $[\alpha k]$ denotes the maximal integer that is smaller than or equal to αk.

In connection with this definition, let us note the following fact:

If $\operatorname{ord} P$ is strictly smaller than 1, then $\exp P$ $(=\sum_{j=0}^{\infty} P^j/j!)$ belongs to $M_r(\mathscr{D}_X^\infty)$.

See S–K–K [6, pp. 438–442] for the proof of this fact.

Now, let R_l $(l=1,\dots,d)$ be in $M_r(\mathscr{D}_X)$ and let $\mathscr{I}$ denote the left $\mathscr{D}_X$-module $\sum_{l=1}^d \mathscr{D}_X^r R_l$. In what follows $\tilde{\mathscr{D}}$ and $\tilde{\mathscr{D}}^\infty$ denote $\mathscr{O}_{\mathbb{C}} \hat{\otimes} \mathscr{D}_X$ and $\mathscr{O}_{\mathbb{C}} \hat{\otimes} \mathscr{D}_X^\infty$, respectively. They are, by the definition (S–K–K [6, p. 418]), sub-rings of $\mathscr{D}_{\mathbb{C}\times X}^\infty$. Accordingly, let $\tilde{\mathscr{I}}$ and $\tilde{\mathscr{I}}^\infty$ denote $\tilde{\mathscr{D}}\mathscr{I}$ and $\tilde{\mathscr{D}}^\infty\mathscr{I}$, respectively. Since $\hat{\otimes}$ is an exact functor, $\tilde{\mathscr{I}} = \mathscr{O}_{\mathbb{C}} \hat{\otimes} \mathscr{I}$ and $\tilde{\mathscr{I}}^\infty = \mathscr{O}_{\mathbb{C}} \hat{\otimes} (\mathscr{D}_X^\infty \otimes_{\mathscr{D}_X} \mathscr{I})$ hold.

LEMMA 1.2. *Let P be in $M_r(\mathscr{D}_X)$ and suppose that it satisfies the following conditions*:

$$\mathscr{I} P \subset \mathscr{I}, \tag{1.2}$$

$$\operatorname{ord} P < 1. \tag{1.3}$$

Then, for any $S(z)$ $(z \in \mathbb{C})$ in $\tilde{\mathscr{I}}^\infty$, we have

$$S(z) \exp(zP) \in \tilde{\mathscr{I}}^\infty. \tag{1.4}$$

Proof. Define $\mathscr{I}(m)$ by $\mathscr{D}(m)^r \cap \mathscr{I}$. Since $\{\mathscr{I}(m)\}_{m\in\mathbb{N}}$ is a good filtration of $\mathscr{I}$,

$$\mathscr{I}(m) = \mathscr{D}(m-m_0)\, \mathscr{I}(m_0) \qquad (m \geqslant m_0) \tag{1.5}$$

holds for sufficiently large m_0. On the other hand, it follows from the definition that there exist constants α $(0 \leqslant \alpha < 1)$ and c such that

$$\text{comp-ord}\, P^k \leqslant [\alpha k] + c \tag{1.6}$$

holds for every non-negative integer k. Hence (1.2) entails

$$\mathscr{I}(m_0)\, P^k \subset \mathscr{I}(m_0 + [\alpha k] + c). \tag{1.7}$$

Then we see from (1.5)

$$\mathscr{I}(m_0)\, P^k \subset \mathscr{D}([\alpha k] + c)\, \mathscr{I}(m_0). \tag{1.8}$$

Let us now choose a system $\{R'_l\}_{l=1}^{d'}$ of generators of $\mathcal{J}(m_0)$ and define a matrix $\mathbf{R}$ by

$$\begin{pmatrix} R'_1 \\ \vdots \\ R'_{d'} \end{pmatrix}.$$

Then there exists a matrix $P(k)$ which satisfies the following condition:

$$\begin{pmatrix} \mathbf{R} \\ \mathbf{R}P \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{R}P^{k-1} \end{pmatrix} P = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ P(k) & 0 & 0 & & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R} \\ \mathbf{R}P \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{R}P^{k-1} \end{pmatrix}. \tag{1.9}$$

In what follows, we denote by $\tilde{P}(k)$ the big matrix in (1.9).

Now, (1.8) entails

$$\text{comp-ord}\, P(k) \leqslant [ak] + c. \tag{1.10}$$

On the other hand, we find

$$\tilde{P}(k)^k = \begin{pmatrix} P(k) & & \\ & \ddots & \\ & & P(k) \end{pmatrix}. \tag{1.11}$$

Therefore we obtain

$$\text{comp-ord}\, \tilde{P}(k)^{nk+j} \leqslant ([ak] + c)n + ([ak] + c)j, \tag{1.12}$$

where n is an arbitrary positive integer and j is a non-negative integer smaller than k. Hence, for every non-negative integer p,

$$\text{comp-ord}\, \tilde{P}(k)^p \leqslant ap + pc/k + ([ak] + c)(k-1) \tag{1.13}$$

holds. Since a is strictly smaller than 1, there exists k_0 such that

$$a + c/k_0 < 1 \tag{1.14}$$

holds. Then it follows from the definition that

$$\text{ord}\, \tilde{P}(k_0) < 1 \tag{1.15}$$

holds. This implies

$$\exp(z\tilde{P}(k_0)) \in \tilde{\mathscr{D}}^{\infty} \qquad (z \in \mathbb{C}). \tag{1.16}$$

Furthermore (1.9) entails

$$\tilde{R}(k_0)\exp(zP) = \exp(z\tilde{P}(k_0))\,\tilde{R}(k_0), \tag{1.17}$$

where $\tilde{R}(k_0)$ denotes

$$\begin{pmatrix} \mathbf{R} \\ \mathbf{R}P \\ \vdots \\ \mathbf{R}P^{k_0-1} \end{pmatrix}.$$

Since $R'_l P^j$ is contained in $\mathscr{D}([aj]+c)\mathscr{J}(m_0)$, (1.17) and (1.16) imply

$$R'_l \exp(zP) \in \tilde{\mathscr{J}}^{\infty} \tag{1.18}$$

for $l = 1,..., d'$. Since $\tilde{\mathscr{J}}^{\infty}$ is also generated by $\{R'_l\}_{l=1}^{d'}$ as a $\tilde{\mathscr{D}}^{\infty}$-module, (1.18) proves the required result. Q.E.D.

PROPOSITION 1.3. *Let P and Q be in $M_r(\mathscr{D}_X)$ and suppose that they satisfy conditions* (1.2) *and* (1.3). *Further suppose that*

$$[P, Q] \equiv cI_r{}^{2} \qquad \operatorname{mod} \mathscr{J} \tag{1.19}$$

holds for some complex number c. Then, for any complex number z, we have

$$\exp(zP)\,Q\exp(-zP) \equiv Q + cz \qquad \operatorname{mod} \tilde{\mathscr{J}}^{\infty}. \tag{1.20}$$

Proof. Let S, $F(z)$, and $G(z)$ denote $[P, Q] - c$, $\exp(zP)Q\exp(-zP)$, and $F(z) - (Q + cz)$, respectively. Then we have

$$\begin{aligned} \frac{\partial G(z)}{\partial z} &= \frac{\partial F(z)}{\partial z} - c \\ &= [P, F(z)] - c \\ &= [P, F(z) - Q - cz] + [P, Q + cz] - c \\ &= [P, G(z)] + S. \end{aligned} \tag{1.21}$$

It is also clear that

$$G(0) = 0 \tag{1.22}$$

[2] Here I_r denotes the $r \times r$ identity matrix. In what follows we abbreviate cI_r to c for simplicity.

holds. Let us now denote $\exp(-zP)\,G(z)\exp(zP)$ by $H(z)$. Then it follows from (1.21) that

$$\begin{aligned}
\frac{\partial}{\partial z}H(z) &= -\exp(-zP)\,PG(z)\exp(zP) \\
&\quad+\exp(-zP)\frac{\partial G(z)}{\partial z}\exp(zP) \\
&\quad+\exp(-zP)\,G(z)P\exp(zP) \\
&= \exp(-zP)\left(\frac{\partial G(z)}{\partial z} - [P, G(z)]\right)\exp(zP) \\
&= \exp(-zP)\,S\exp(zP).
\end{aligned} \tag{1.23}$$

Since $\exp(-zP)$ belongs to $\tilde{\mathscr{D}}^{\infty}$ by assumption (1.3), Lemma 1.2 implies

$$\frac{\partial H(z)}{\partial z} \in \tilde{\mathscr{I}}^{\infty}. \tag{1.24}$$

Hence $\partial H(z)/\partial z$ has the form

$$\sum_{l=1}^{d} h_l(z)\,R_l, \tag{1.25}$$

with $h_l(z)$ $(l=1,\ldots,d)$ belonging to $\tilde{\mathscr{D}}_X^{\infty}$, where $\{R_l\}_{l=1}^{d}$ are the system of generators of $\mathscr{I}$. Then, by defining $I_l(z)$ by $\int_0^z h_l(w)\,dw$, we find

$$\frac{\partial}{\partial z}\left(H(z) - \sum_{l=1}^{d} I_l(z)\,R_l\right) = 0. \tag{1.26}$$

It also follows from the definition that

$$H(0) - \sum_{l=1}^{d} I_l(0)\,R_l = H(0) = 0 \tag{1.27}$$

holds. Therefore we conclude that $H(z)$ belongs to $\tilde{\mathscr{I}}^{\infty}$. It then follows from the definition of $H(z)$ and Lemma 1.2 that $G(z)$ belongs to $\tilde{\mathscr{I}}^{\infty}$. This proves the required relation (1.20). Q.E.D.

THEOREM 1.4. *Let P and Q be the same as in Proposition* 1.3. *Then, for any complex number z, we have*

$$\exp(zP)\exp(zQ) \equiv \exp\left(z(P+Q) + \frac{cz^2}{2}\right) \mod \tilde{\mathscr{I}}^{\infty}, \tag{1.28}$$

in particular,

$$\exp P \exp Q \equiv \exp\left(P + Q + \frac{c}{2}\right) \qquad \mathrm{mod}\ \mathscr{D}_X^\infty \otimes_{\mathscr{D}_X} \mathscr{I}. \tag{1.29}$$

Proof. Let $\Phi(z, P, Q, c)$ denote $\exp(zP)\exp(zQ)\exp(-cz^2/2)$. Then we have

$$\begin{aligned}\frac{\partial\Phi}{\partial z} &= P\Phi + \exp(zP)\,Q\exp(zQ)\exp\left(-\frac{cz^2}{2}\right) - cz\Phi \\ &= P\Phi + \exp(zP)\,Q\exp(-zP)\,\Phi - cz\Phi.\end{aligned} \tag{1.30}$$

Hence, by the aid of Proposition 1.3, we find

$$\frac{\partial\Phi}{\partial z} = (P + Q + S(z))\,\Phi \tag{1.31}$$

with $S(z)$ in $\tilde{\mathscr{I}}^\infty$. Then Lemma 1.2 guarantees that

$$\frac{\partial\Phi}{\partial z} \equiv (P + Q)\,\Phi \qquad \mathrm{mod}\ \tilde{\mathscr{I}}^\infty. \tag{1.32}$$

Now let us consider $\Psi =_{\mathrm{def}} \Phi - \exp(z(P + Q))$. It then follows from (1.32) that

$$\frac{\partial\Psi}{\partial z} \equiv 0 \qquad \mathrm{mod}\ \tilde{\mathscr{I}}^\infty \tag{1.33}$$

holds. Furthermore $\Psi(0) = 0$ holds. Hence, by using the same reasoning as was used at the end of the proof of Proposition 1.3, we conclude that $\Psi(z)$ belongs to $\tilde{\mathscr{I}}^\infty$. Thus we have shown

$$\exp(zP)\exp(zQ)\exp\left(-\frac{cz^2}{2}\right) \equiv \exp(z(P + Q)) \qquad \mathrm{mod}\ \tilde{\mathscr{I}}^\infty.$$

This immediately implies (1.28) and (1.29). Q.E.D.

2. Theta Functions and Jacobi Functions

Let X be an open subset of $\mathbb{C}^m$ and let $t = (t_1,..., t_m)$ denote a coordinate system on it. Let $\mathscr{N}$ be a coherent $\mathscr{D}_X$-module $\mathscr{D}_X^r/\mathscr{I}$, where $\mathscr{I}$ has the form $(\sum_{l=1}^d \mathscr{D}_X^r R_l)$ with R_l $(l = 1,..., d)$ in $M_r(\mathscr{D}_X)$.

DEFINITION 2.1 (Jacobi structure). Let P be a set of matrices P_j ($j = 1,\dots, 2n$) of linear differential operators on X. If P satisfies the following conditions (2.1), (2.2) and (2.3), we call it a Jacobi structure (with respect to $\mathcal{N}$).

$$P_j \in M_r(\mathscr{D}_X) \text{ and } \mathscr{I}P_j \subset \mathscr{I} \text{ holds for } j = 1,\dots, 2n. \tag{2.1}$$

$$\text{For any } (c_1,\dots, c_{2n}) \text{ in } \mathbb{C}^{2n}, \operatorname{ord}\left(\sum_{j=1}^{2n} c_j P_j\right) < 1. \tag{2.2}$$

There exists a matrix $E = (e_{jk})$ in $SL(2n; \mathbb{Z})$
which satisfies the following relation: (2.3)

$$[P_j, P_k] \equiv -2\pi\sqrt{-1}\, e_{jk} \qquad \operatorname{mod} \mathscr{I} \qquad (1 \leqslant j, k \leqslant 2n).$$

Remark 2.2. If there is no fear of confusions, we often omit the phrase "with respect to $\mathcal{N}$."

Remark 2.3. We call the matrix E the structure matrix of the Jacobi structure P.

DEFINITION 2.4. Let P be a Jacobi structure with respect to $\mathcal{N}$. If an r-tuple of holomorphic function $h(t)$ on X satisfies the following Eqs. (2.4) and (2.5) with some $c = (c_1,\dots, c_{2n}) \in \mathbb{C}^{2n}$, we call $h(t)$ a Jacobi function.

$$(\exp P_j)\, h(t) = c_j h(t) \qquad (j = 1,\dots, 2n). \tag{2.4}$$

$$R_l h(t) = 0 \qquad (l = 1,\dots, d). \tag{2.5}$$

The set of all Jacobi functions is denoted by $J(P, c)$.

Remark 2.5. (i) By using Theorem 1.4, we obtain the following relation (2.6) from condition (2.3):

$$\begin{aligned} \exp P_j \exp P_k &\equiv \exp(P_j + P_k - \pi\sqrt{-1}\, e_{jk}) \\ &\equiv \exp P_k \exp P_j \exp(-2\pi\sqrt{-1}\, e_{jk}) \\ &= \exp P_k \exp P_j \qquad \operatorname{mod} \mathscr{D}^\infty \otimes_{\mathscr{D}} \mathscr{I}. \end{aligned} \tag{2.6}$$

Hence, considering the simultaneous eigenvalue problem (2.4) with the subsidiary condition (2.5) makes sense.

(ii) Condition (2.2) guarantees that $\exp P_j$ belongs to $\mathscr{D}_X^\infty$. Hence the notion of Jacobi functions is a local one.

DEFINITION 2.6. Let P be a Jacobi structure with respect to $\mathcal{N}$. If an r-vector of hyperfunctions $\vartheta(x \mid t)$ on $\mathbb{R}_x^{2n} \times X$ satisfies the following relations (2.7), (2.8), (2.9) and (2.10), then we call it a theta function (associated with P).

$$\left(\frac{\partial}{\partial x_j} - \pi\sqrt{-1}(Ex)_j\right)\vartheta(x \mid t) = P_j\vartheta(x \mid t),^3 \qquad j = 1,..., 2n, \tag{2.7}$$

$$R_l\vartheta(x \mid t) = 0, \qquad l = 1,..., d. \tag{2.8}$$

$$\frac{\partial}{\partial \bar{t}_p}\vartheta(x \mid t) = 0,^4 \qquad p = 1,..., m. \tag{2.9}$$

For each ν in $\mathbb{Z}^{2n}$, there exists a constant $c(\nu)$ so that

$$\vartheta(x + \nu \mid t) = c(\nu)\exp(\pi\sqrt{-1}\langle E\nu, x\rangle)\,\vartheta(x \mid t) \tag{2.10}$$

holds.

Remark 2.7. We call condition (2.10) the quasi-periodicity condition after the terminology used for the classical elliptic theta functions.

Remark 2.8. Condition (2.3) guarantees that Eqs. (2.7) and (2.8) are compatible.

Remark 2.9. Since the system of differential equations (2.7), (2.8) and (2.9) is elliptic, a theta function discussed here is necessarily real analytic. Furthermore, as our later argument will show, it can be extended as a holomorphic function on $\mathbb{C}^{2n} \times X$.

Now we list the results which clarify the relations between Jacobi functions and theta function.

THEOREM 2.10. *Let P be a Jacobi structure with respect to $\mathscr{N}$. Then we have the following*:

(i) *If $h(t)$ belongs to $J(P, c)$, then*

$$\varphi(x \mid t) \underset{\text{def}}{=} \exp\left(\sum_{j=1}^{2n} x_jP_j\right) h(t)$$

is a theta function with

$$c(\nu) = (-1)^{\sum_{1 \leqslant j<k \leqslant 2n}\nu_j\nu_k e_{jk}}\, c_1^{\nu_1} \cdots c_{2n}^{\nu_{2n}}. \tag{2.11}$$

Furthermore, $\varphi(0 \mid t) = h(t)$ holds.

(ii) *If $\vartheta(x \mid t)$ is a theta function, then $\vartheta(0 \mid t)$ belongs to $J(P, c)$ with c_j $(1 \leqslant j \leqslant 2n)$ being given by $c((0,..., 0, \overset{j}{1}, 0,..., 0))$. Furthermore, $\vartheta(x \mid t) = \exp(\sum_{j=1}^{2n} x_jP_j)\,\vartheta(0 \mid t)$ holds.*

[3] Here and in what follows, $(Ex)_j$ denotes the jth component of the vector Ex, namely, $(Ex)_j = \sum_{k=1}^{2n} e_{jk}x_k$.

[4] Here $\partial/\partial\bar{t}_p$ denotes the Cauchy–Riemann operator, namely, $\partial/\partial\bar{t}_p = 1/2[\partial/\partial\sigma_p + \sqrt{-1}\,\partial/\partial\tau_p]$, where $\sigma_p = \operatorname{Re} t_p$ and $\tau_p = \operatorname{Im} t_p$.

Remark 2.11. This theorem tells us that a Jacobi function may be called a theta zero-value on the analogy of the terminology used for elliptic theta functions.

Proof. (i) It is clear that $\varphi(x \mid t)$ defined above satisfies (2.8) and (2.9). Hence it follows from Theorem 1.4 that we have

$$\begin{aligned} \varphi(x \mid t) &= \exp(x_j P_j) \exp\left(\sum_{k \neq j} x_k P_k\right) \exp\left(-\frac{1}{2}\left[x_j P_j, \sum_{k \neq j} x_k P_k\right]\right) h(t) \\ &= \exp(x_j P_j) \exp\left(\sum_{k \neq j} x_k P_k\right) \exp\left(\pi\sqrt{-1}\, x_j \left(\sum_{k \neq j} e_{jk} x_k\right)\right) h(t). \end{aligned} \tag{2.12}$$

Since $e_{jj} = 0$, (2.12) entails

$$\begin{aligned} &\left(\frac{\partial}{\partial x_j} - \pi\sqrt{-1}(Ex)_j\right) \varphi(x \mid t) \\ &\qquad = P_j \varphi + \pi\sqrt{-1}\left(\sum_{k \neq j} e_{jk} x_k\right)\varphi - \pi\sqrt{-1}(Ex)_j \varphi \\ &\qquad = P_j \varphi. \end{aligned} \tag{2.13}$$

Thus we have verified that φ satisfies (2.7). By exactly the same reasoning, we see that

$$\varphi(x + a \mid t) = \exp(\pi\sqrt{-1}\langle Ea, x\rangle) \exp\left(\sum_{j=1}^{2n} x_j P_j\right) \exp\left(\sum_{j=1}^{2n} a_j P_j\right) h(t) \tag{2.14}$$

holds for every $a =_{\text{def}} (a_1, \ldots, a_{2n})$ in $\mathbb{R}^{2n}$. On the other hand, again by Theorem 1.4 we obtain from (2.4) and (2.5)

$$\exp\left(\sum_{j=1}^{2n} \nu_j P_j\right) h(t) = \exp\left(\pi\sqrt{-1}\left(\sum_{1 \leq j < k \leq 2n} \nu_j \nu_k e_{jk}\right)\right) \prod_{j=1}^{2n} c_j^{\nu_j} h(t) \tag{2.15}$$

for any $\nu =_{\text{def}} (\nu_1, \ldots, \nu_{2n})$ in $\mathbb{Z}^{2n}$. Hence, by choosing a in (2.14) to be in the lattice $\mathbb{Z}^{2n}$, we find

$$\varphi(x + \nu \mid t) = (-1)^{\sum_{1 \leq j < k \leq 2n} \nu_j \nu_k e_{jk}} \prod_{j=1}^{2n} c_j^{\nu_j} \exp(\pi\sqrt{-1}\langle E\nu, x\rangle)\, \varphi(x \mid t) \qquad (\nu \in \mathbb{Z}^{2n}). \tag{2.16}$$

Therefore $\varphi(x \mid t)$ satisfies (2.10) with $c(\nu)$ being given by (2.11). Since it is clear that $\varphi(0 \mid t) = h(t)$, this completes the proof of (i).

(ii) Since R_l and $\partial/\partial \bar{t}_p$ are differential operators in t-variables, $R_l \vartheta(0 \mid t) = 0$ and $(\partial/\partial \bar{t}_p)\, \vartheta(0 \mid t) = 0$ hold for any l and any p. Hence we can use the same reasoning as in the proof of (i) to find that $\tilde{\vartheta}(x \mid t) =_{\mathrm{def}} \exp(\sum_{j=1}^{2n} x_j P_j)\, \vartheta(0 \mid t)$ satisfies Eqs. (2.7), (2.8) and (2.9). Furthermore $\tilde{\vartheta}(x \mid t)$ is holomorphic on $\mathbb{C}^{2n} \times X$. Since $\tilde{\vartheta}(0 \mid t) = \vartheta(0 \mid t)$ holds by the definition, and since $\vartheta(x \mid t)$ is also analytic (Remark 2.9), the local uniqueness assertion in the Cauchy–Kovalevsky theorem guarantees that $\tilde{\vartheta}(x \mid t) = \vartheta(x \mid t)$ holds on $\mathbb{R}^{2n} \times X$. In particular, we have

$$\exp\left(\sum_{j=1}^{2n} \nu_j P_j\right) \vartheta(0 \mid t) = \tilde{\vartheta}(\nu \mid t) = \vartheta(\nu \mid t) \tag{2.17}$$

for each $\mathbf{v} = (\nu_1, \ldots, \nu_{2n})$ in $\mathbb{Z}^{2n}$.

On the other hand, the quasi-periodicity condition asserts

$$\vartheta(\nu \mid t) = c(\nu)\, \vartheta(0 \mid t). \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$(\exp P_j)\, \vartheta(0 \mid t) = c(\mathbf{v}_j)\, \vartheta(0 \mid t), \qquad 1 \leqslant j \leqslant 2n,$$

where $\mathbf{v}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$. Thus we have verified that $\vartheta(0 \mid t)$ belongs to $J(P, c)$ with c_j being given by $c(\mathbf{v}_j)$. This completes the proof of (ii). Q.E.D.

Now that we have established the correspondence between Jacobi functions and theta functions, we embark on the proof of the finite dimensionality of $J(P, c)$. As we mentioned in the introduction, this is an analogue for theta functions of the classical Hamburger theorem for the ζ-function of Riemann. To prove the desired result, however, we need to require that the Jacobi structure in question should satisfy condition (2.21) stated below. In order to state the condition we prepare some notations:

First let us note that condition (2.1) makes it possible to define an endomorphism Φ_j of $\mathscr{N}$ by assigning $QP_j u$ to Qu, where u is a generator of $\mathscr{N}$ and Q belongs to $\mathscr{D}_X^r$. In what follows, we denote by u^{Φ_j} the image of u by Φ_j; that is, Φ_j is, by definition, to act upon $\mathscr{N}$ from the right. By this convention, $P_j P_k u$ corresponds to $u^{\Phi_j \Phi_k}$. Let W denote the $\mathbb{Z}$-module $\bigoplus_{j=1}^{2n} \mathbb{Z} \Phi_j$. Since it follows from (2.3) that

$$\Phi_j \Phi_k - \Phi_k \Phi_j = -2\pi \sqrt{-1}\, e_{jk} \qquad (1 \leqslant j, k \leqslant 2n) \tag{2.19}$$

holds, we can define a skew-symmetric inner product $\langle \Phi, \Phi' \rangle$ of Φ and Φ' in W by $\Phi\Phi' - \Phi'\Phi$. For a subspace V of W, $V^\perp$ denotes the subspace

$\{\Psi \in W;\ \langle \Phi, \Psi \rangle = 0$ holds for every Φ in $V\}$. Let $V_{\mathbb{Q}}$ and $(V^{\perp})_{\mathbb{Q}}$ denote the vector spaces $\mathbb{Q} \otimes V$ and $\mathbb{Q} \otimes V^{\perp}$, respectively. Since the inner product introduced in the above is non-degenerate, we see

$$\dim V_{\mathbb{Q}} + \dim (V^{\perp})_{\mathbb{Q}} = 2n.$$

Hence, if $V = V^{\perp}$ holds, then $\dim V_{\mathbb{Q}} = n$ holds. In this case we say that V is Lagrangian. In what follows, for Q in $M_r(\mathscr{D}_X)$ which is equal to $\sum_{j=1}^{2n} c_j P_j$ modulo $\mathscr{I}$ with complex numbers c_j, we let $\Phi(Q)$ denote $\sum_{j=1}^{2n} c_j \Phi_j$.

DEFINITION 2.12. A partial Jacobi system $\mathscr{L}(V)$ associated with a Lagrangian subspace V of W is, by definition, the following $\mathscr{D}_X$-module:

$$\mathscr{D}_X^r \Big/ \Big(\sum_{\Phi(Q) \in V} \mathscr{D}_X^r Q + \mathscr{I} \Big). \tag{2.20}$$

DEFINITION 2.13. A pair $(P, \mathscr{I})$ is said to be maximal if there exists a Lagrangian subspace V of W such that the associated partial Jacobi system $\mathscr{L}(V)$ is a holonomic system.[5]

Now, the condition that guarantees the finite dimensionality of $J(P, c)$ is the following:

$$\text{The pair } (P, \mathscr{I}) \text{ is maximal.} \tag{2.21}$$

In fact, assuming this condition, we have the following

THEOREM 2.14. *Let P be a Jacobi structure with respect to a $\mathscr{D}_X$-module $\mathscr{N} = \mathscr{D}_X^r/\mathscr{I}$. Suppose that the pair $(P, \mathscr{I})$ is maximal. Then,* $\dim_{\mathbb{C}} J(P, c)$ *is finite for every c in $\mathbb{C}^{2n}$. Furthermore, it is independent of c, if c belongs to $(\mathbb{C} - \{0\})^{2n}$.*

Remark 2.16. Since $\exp(-P_j)\exp(P_j)$ is the identity operator, $J(P, c)$ consists of only zero, if some $c_j = 0$. This is the reason why the set $(\mathbb{C} - \{0\})^{2n}$ appears in the theorem.

Proof. Let us first show the finite dimensionality of $J(P, c)$. As we have noticed in Remark 2.15 above, $\dim J(P, c) = 0$ if some $c_j = 0$. Therefore there is nothing to prove in this case. Hence we assume

$$c_j \neq 0 \qquad (j = 1, \dots, 2n). \tag{2.22}$$

Now, by virtue of Theorem 2.10, it suffices to show the finite dimen-

[5] The terminology "maximally overdetermined system" is used in S–K–K [6] etc. See [4] and the references cited there for the theory of holonomic systems.

sionality of the space of thata functions, assuming that the constant $c(\nu)$ in condition (2.10) is given by (2.11).

As $(P, \mathscr{J})$ is maximal by the assumption, there exists a Lagrangian subspace $V = \bigoplus_{j=n+1}^{2n} \mathbb{Z}\Phi_j$ of W such that the associated partial Jacobi system $\mathscr{L}(V)$ is holonomic. We first show that we may assume without loss of generality that E has the form

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Since ${}^tE = -E$ and $\det E = 1$ hold by the definition, the theory of elementary divisors tells us that there exists a matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant 2n}$ which satisfies the following conditions:

$$a_{ij} \in \mathbb{Z} \qquad (1 \leqslant i, j \leqslant 2n), \qquad |\det A| = 1, \tag{2.23}$$

$$AE\,{}^tA = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}, \tag{2.24}$$

$$\sum_{j=n+1}^{2n} a_{ij}\Phi_j = 0 \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad \sum_{j=n+1}^{2n} a_{ij}\Phi_j \in V \quad (n+1 \leqslant i \leqslant 2n). \tag{2.25}$$

Let $\tilde{E} = (\tilde{e}_{ij})_{1 \leqslant i,j \leqslant 2n}$ and $B = (b_{ij})_{1 \leqslant i,j \leqslant 2n}$ denote $AE\,{}^tA$ and ${}^tA^{-1}$, respectively. Note that (2.23) guarantees that every b_{ij} is an integer. Let us now introduce a new coordinate $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{2n})$ and a new Jacobi structure $\tilde{P} = (\tilde{P}_1, \ldots, \tilde{P}_{2n})$ by

$$\tilde{x}_i = \sum_{j=1}^{2n} b_{ij} x_j \qquad (i = 1, \ldots, 2n) \tag{2.26}$$

and

$$\tilde{P}_i = \sum_{j=1}^{2n} a_{ij} P_j \qquad (i = 1, \ldots, 2n), \tag{2.27}$$

respectively. We now show that, if we define $\tilde{\vartheta}(\tilde{x} \mid t)$ by $\vartheta({}^tA\tilde{x} \mid t)$, then $\tilde{\vartheta}(\tilde{x} \mid t)$ is a theta function associated with the Jacobi structure $\tilde{P}$. Since $\tilde{E}\tilde{x} = AEx$ holds by the definition, we find the following (2.28) from (2.7):

$$\left(\frac{\partial}{\partial \tilde{x}_i} - \pi\sqrt{-1}(\tilde{E}\tilde{x})_i\right) \tilde{\vartheta}(\tilde{x} \mid t)$$

$$= \left[\sum_j a_{ij}\left(\frac{\partial}{\partial x_j} - \pi\sqrt{-1}(Ex)_j\right) \vartheta(x \mid t)\right]\Bigg|_{x = A\tilde{x}}$$

$$= \left[\sum_j a_{ij} P_j \left(t, \frac{\partial}{\partial t}\right) \vartheta(x \mid t)\right]\Bigg|_{x = A\tilde{x}}$$

$$= \tilde{P}_i \left(t, \frac{\partial}{\partial t}\right) \vartheta(x \mid t) \Bigg|_{x = A\tilde{x}}$$

$$= \tilde{P}_i \left(t, \frac{\partial}{\partial t}\right) \tilde{\vartheta}(\tilde{x} \mid t) \qquad (i = 1,..., 2n). \tag{2.28}$$

Since $R_l \tilde{\vartheta}(\tilde{x} \mid t) = 0$ $(1 \leqslant l \leqslant d)$ and $(\partial/\partial \bar{t}_p)\, \tilde{\vartheta}(\tilde{x} \mid t) = 0$ $(1 \leqslant p \leqslant m)$ clearly hold, it now suffices to show the quasi-periodicity of $\tilde{\vartheta}(\tilde{x} \mid t)$. Since every component of A is an integer, every component of ${}^tA\mu$ is also an integer, if so is every component of the column vector $\mu = {}^t(\mu_1,..., \mu_{2n})$. Hence, by using (2.10), we find

$$\begin{aligned} \tilde{\vartheta}(\tilde{x} + \mu \mid t) &= \vartheta({}^tA\tilde{x} + {}^tA\mu \mid t) \\ &= c({}^tA\mu) \exp(\pi \sqrt{-1} \langle E\, {}^tA\mu, {}^tA\tilde{x}\rangle)\, \vartheta({}^tA\tilde{x} \mid t) \\ &= c({}^tA\mu) \exp(\pi \sqrt{-1} \langle \tilde{E}\mu, \tilde{x}\rangle)\, \tilde{\vartheta}(\tilde{x} \mid t). \end{aligned}$$

This means that $\tilde{\vartheta}(\tilde{x} \mid t)$ also satisfies the quasi-periodicity condition. Therefore $\tilde{\vartheta}(\tilde{x} \mid t)$ is a theta function associated with the Jacobi structure $\tilde{P}$. Since $\tilde{\vartheta}(0 \mid t) = \vartheta(0 \mid t)$ holds by the definition, and since $\tilde{\vartheta}(\tilde{x} \mid t)$ is uniquely determined by $\tilde{\vartheta}(0 \mid t)$ (Theorem 2.10(ii)), it suffices to show the finite dimensionality of the space of theta functions associated with $\tilde{P}$. Note that (2.25) guarantees that $(\tilde{P}, \mathscr{J})$ is maximal. Thus we may assume without loss of generality that E has the form

$$\begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}.$$

Now, let us choose constants a_j $(j = 1,..., 2n)$ so that the following holds:

$$\exp(\pi \sqrt{-1}\, a_j) = c_j \qquad (j = 1,..., 2n). \tag{2.29}$$

Thanks to assumption (2.22), such constants a_j really exist. Using these constants a_j, we define an analytic function $\eta(x \mid t)$ by

$$\exp\left(\pi \sqrt{-1} \left(\sum_{j=1}^{n} (x_j + a_j)(x_{j+n} + a_{j+n})\right)\right) \vartheta(x \mid t).$$

In what follows, let x' etc. and x'' etc. denote, respectively, $(x_1,..., x_n)$ etc. and $(x_{n+1},..., x_{2n})$ etc. For the brevity of notation, we also denote $\sum_{j=1}^{n} x_j x_{j+n}$ etc. by $\langle x', x''\rangle$ etc.

We now show that the quasi-periodicity condition (2.10) entails the periodicity of η with respect to x''-variables. In fact, using the fact that $c(\nu)$ is given by (2.11) and the fact that

$$E = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix},$$

we find, for each $\nu = (\nu', \nu'')$ in $\mathbb{Z}^{2n}$, the following relation:

$$\begin{aligned}
&\eta(x' + \nu', x'' + \nu'' \mid t) \\
&\quad = \exp(\pi \sqrt{-1}\langle x' + \nu' + a', x'' + \nu'' + a''\rangle)\, \vartheta(x + \nu \mid t) \\
&\quad = \exp(\pi \sqrt{-1}\langle x' + \nu' + a', x'' + \nu'' + a''\rangle) \\
&\qquad \times (-1)^{\langle \nu', \nu''\rangle} \prod_{j=1}^{2n} c_j^{\nu_j} \exp(\pi \sqrt{-1}(\langle -\nu'', x'\rangle + \langle \nu', x''\rangle))\, \vartheta(x \mid t) \\
&\quad = \exp(2\pi \sqrt{-1}\langle \nu', x''\rangle) \exp(\pi \sqrt{-1}\langle x' + a', x'' + a''\rangle)\, \vartheta(x \mid t) \\
&\quad = \exp(2\pi \sqrt{-1}\langle \nu', x''\rangle)\, \eta(x \mid t).
\end{aligned} \tag{2.30}$$

In particular, we have

$$\eta(x', x'' + \nu'' \mid t) = \eta(x', x'' \mid t) \tag{2.31}$$

for any ν'' in $\mathbb{Z}^n$.

Thus, $\eta(x', x'' \mid t)$ is a real analytic function periodic in x''. Hence, it has the form

$$\sum_{\mu \in \mathbb{Z}^n} f_\mu(x' \mid t) \exp(2\pi \sqrt{-1}\langle \mu, x''\rangle) \tag{2.32}$$

with $f_\mu(x' \mid t)$ being given by

$$\int_0^1 \cdots \int_0^1 \eta(x', x'' \mid t) \exp(-2\pi \sqrt{-1}\langle \mu, x''\rangle)\, dx''. \tag{2.33}$$

Here and in what follows, dx'' denotes $\prod_{j=n+1}^{2n} dx_j$. Now, by (2.30), we have

$$\eta(x' + \rho, x'' \mid t) = \exp(2\pi i\langle \rho, x''\rangle)\, \eta(x', x'' \mid t) \tag{2.34}$$

for any ρ in $\mathbb{Z}^n$. Hence it follows from (2.33) that

$$f_\mu(x'+\rho \mid t) = f_{\mu-\rho}(x' \mid t) \tag{2.35}$$

holds for every μ in $\mathbb{Z}^n$. This means that $\eta(x \mid t)$, and hence $\vartheta(x \mid t)$, is uniquely determined by the function $f_0(x' \mid t)$ globally defined on $\mathbb{R}^n_{x'} \times X$.

Now, thanks to the particular form of the matrix E, Eqs. (2.7) imply the following:

$$\begin{aligned}
\frac{\partial}{\partial x_j} f_0(x' \mid t) &= \frac{\partial}{\partial x_j} \int_0^1 \cdots \int_0^1 \eta(x', x'' \mid t)\, dx'' \\
&= \frac{\partial}{\partial x_j} \int_0^1 \cdots \int_0^1 \exp(\pi\sqrt{-1}\langle x'+a', x''+a''\rangle)\, \vartheta(x', x'' \mid t)\, dx'' \\
&= \int_0^1 \cdots \int_0^1 \Big[\pi\sqrt{-1}(x_{j+n}+a_{j+n}) - \pi\sqrt{-1}\, x_{j+n} \\
&\qquad + P_j\left(t, \frac{\partial}{\partial t}\right)\Big]\eta(x', x'' \mid t)\, dx'' \\
&= \left[\pi\sqrt{-1}\, a_{j+n} + P_j\left(t, \frac{\partial}{\partial t}\right)\right] f_0(x' \mid t) \qquad (j=1,\dots,n)
\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}
&2\pi\sqrt{-1}\left(x_j + \frac{a_j}{2}\right) f_0(x' \mid t) \\
&\quad = \int_0^1 \cdots \int_0^1 \frac{\partial}{\partial x_{j+n}} \eta(x', x'' \mid t)\, dx'' \\
&\quad\quad - \int_0^1 \cdots \int_0^1 P_{j+n}\left(t, \frac{\partial}{\partial t}\right) \eta(x', x'' \mid t)\, dx'' \\
&\quad = -P_{j+n}\left(t, \frac{\partial}{\partial t}\right) f_0(x' \mid t) \qquad (j=1,\dots,n).
\end{aligned} \tag{2.37}$$

It is clear that $R_l(t, \partial/\partial t) f_0(x' \mid t) = 0$ and $(\partial/\partial \bar{t}_p) f_0(x' \mid t) = 0$ hold for any l and p. Since $\mathscr{L}(V) = \mathscr{L}(\bigoplus_{j=n+1}^{2n} \mathbb{Z}\Phi_j)$ is holonomic by the assumption,

$$\dim_{\mathbb{C}} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}(V), \mathscr{O}_X)_p < \infty \tag{2.38}$$

holds for any point p in X. Hence, if we denote by $F_{-a'/2}$ the vector space formed by the collection of possible $f_0(-a'/2 \mid t)$, it follows from (2.37) that $\dim_{\mathbb{C}} F_{-a'/2}$ is finite. Next, by using the fact that $R_l(t, \partial/\partial t) f_0(-a'/2 \mid t) = 0$ $(l=1,\dots,d)$, we conclude from Theorem 1.4 that

$$\varphi_0(x' \mid t) = \exp\left(\sum_{j=1}^{n}\left(x_j + \frac{a_j}{2}\right)\left(\pi\sqrt{-1}\, a_{j+n} + P_j\left(t, \frac{\partial}{\partial t}\right)\right)\right) f_0\left(-\frac{a'}{2} \,\middle|\, t\right)$$

is a solution of the following Eqs. (2.39) considered on $\mathbb{R}^n \times X$:

$$\left(\frac{\partial}{\partial x_j} - \pi\sqrt{-1}\, a_{j+n} - P_j\left(t, \frac{\partial}{\partial t}\right)\right)\varphi_0(x' \mid t) = 0 \qquad (j = 1,\dots, n). \tag{2.39}$$

Here we have used condition (2.2) to guarantee that $\varphi_0(x' \mid t)$ is well-defined over $\mathbb{R}^n \times X$ (actually over $\mathbb{C}^n \times X$).

In view of the uniqueness of the solution to the Cauchy problem for Eqs. (2.39) with the Cauchy data on $\{(x', t) \in \mathbb{R}^n \times X;\ x' = 0\}$, we find that $\varphi_0(x' \mid t) = f_0(x' \mid t)$ holds on $\mathbb{R}^n \times X$. Since $\varphi_0(x' \mid t)$ is uniquely determined by $f_0(-a'/2 \mid t)$, the finite dimensionality of the space $F_{-a'/2}$ implies the finite dimensionality of the space of all possible $f_0(x' \mid t)$. Since we know that $\vartheta(x \mid t)$ is uniquely determined by $f_0(x' \mid t)$, the space of theta functions is finite dimensional. Hence it follows from Theorem 2.10 that $\dim_{\mathbb{C}} J(P, c)$ is finite.

Finally we show that

$$\dim_{\mathbb{C}} J(P, c) = \dim_{\mathbb{C}} J(P, c') \tag{2.40}$$

holds if both c and c' belong to $(\mathbb{C} - \{0\})^{2n}$. Let a_j $(j = 1,\dots, 2n)$ be constants which satisfy

$$\frac{c'_j}{c_j} = \exp(-2\pi\sqrt{-1}(Ea)_j) \qquad (j = 1,\dots, 2n). \tag{2.41}$$

Since c_j is different from 0 and since E is an invertible matrix, such a constant c_j really exists. Then, by using Theorem 1.4, we find the following relation (2.42) for $h(t)$ in $J(P, c)$:

$$\begin{aligned}(\exp P_j)\left(\exp\left(\sum_{k=1}^{2n} a_k P_k\right)\right) h(t) &= \exp\left(-2\pi\sqrt{-1}\left(\sum_{k=1}^{2n} e_{jk} a_k\right)\right)\left(\exp\left(\sum_{k=1}^{2n} a_k P_k\right)\right)(\exp P_j)\, h(t) \\ &= c'_j\left(\exp\left(\sum_{k=1}^{2n} a_k P_k\right)\right) h(t).\end{aligned} \tag{2.42}$$

This means that $\exp(\sum_{k=1}^{2n} a_k P_k)\, h(t)$ belongs to $J(P, c')$. Since $\exp(\sum_{k=1}^{2n} a_k P_k)$ is an invertible linear differential operator, this implies

$$\dim_{\mathbb{C}} J(P, c) \leqslant \dim_{\mathbb{C}} J(P, c'). \tag{2.43}$$

By changing the role of c and c', we find the opposite inequality, and hence (2.40). This completes the proof of the theorem.

Remark 2.16. In the course of the proof of the theorem, we have obtained the following inequality:

$$\dim_{\mathbb{C}} J(P, c) \leqslant \dim_{\mathbb{C}} F_{-a'/2}. \tag{2.44}$$

Our argument also shows that, if we can somehow verify the finite dimensionality of the space of (global) solutions of the system of Eqs. (2.36), (2.37), $R_l f_0 = 0$ and $(\partial/\partial \bar{t}_p) f_0 = 0$ for any l and p, then $\dim_{\mathbb{C}} J(P, c)$ is seen to be finite for every c.

3. Examples

The purpose of this section is to present a recipe by which we can find examples of Jacobi structures. The recipe will make it clear that the introduction of the subsidiary system $\mathscr{N}$ facilitates the construction of Jacobi structures.

To start with, let us consider an analytic function $f_0(x \mid t)$ defined on $\mathbb{R}^n_x \times X$ which satisfies the following conditions (3.1) and (3.2):

$$f_0(x \mid t) \text{ is holomorphic in } t. \tag{3.1}$$

There exist linear differential operators $A_{jj'}(t, \partial/\partial t)$, $B_{kk'}(t, \partial/\partial t)$ and $C_{ll'}(t, \partial/\partial t)$ $(1 \leqslant j, j', k, k', l, l' \leqslant n)$ of order 1 and defined on X which satisfy the following: (3.2)

$$x_j x_{j'} f_0(x \mid t) = A_{jj'} f_0(x \mid t) \qquad (1 \leqslant j, j' \leqslant n), \tag{3.2.a}$$

$$x_k \frac{\partial}{\partial x_{k'}} f_0(x \mid t) = B_{kk'} f_0(x \mid t) \qquad (1 \leqslant k, k' \leqslant n), \tag{3.2.b}$$

$$\frac{\partial^2}{\partial x_l\, \partial x_{l'}} f_0(x \mid t) = C_{ll'} f_0(x \mid t) \qquad (1 \leqslant l, l' \leqslant n). \tag{3.2.c}$$

Let us now consider a $(2n+1)$-column vector $\mathfrak{f}(x \mid t) = {}^t(f_0, x_1 f_0, \ldots, x_n f_0,$

$(\partial/\partial x_1)f_0,\dots,(\partial/\partial x_n)f_0)$. Define $(2n+1)\times(2n+1)$-matrix P_j $(j=1,\dots,2n)$ by

$$P_j = \begin{array}{rc}
 & \begin{array}{cccc} 0 & & n+j & \end{array}\\
\begin{array}{r} \\ \\ \\ j) \\ \\ \\ \\ \\ \\ \end{array} & \left[\begin{array}{cccc} 0 & 0 & 1 & 0\\ B_{j1} & & & \\ \vdots & & & \\ B_{jj}+1 & & & \\ \vdots & & 0 & \\ B_{jn} & & & \\ C_{1j} & & & \\ \vdots & & & \\ C_{nj} & & & \end{array}\right]
\end{array} \qquad (j=1,\dots,n), \tag{3.3.a}$$

$$P_{j+n} = -2\pi\sqrt{-1}\ \begin{array}{c}
\begin{array}{cccc} 0 & & j & \end{array}\\
\left[\begin{array}{cccc} 0 & 0 & 1 & 0\\ A_{j1} & & & \\ \vdots & & & \\ A_{jn} & & & \\ & & 0 & \\ B_{j1} & & & \\ \vdots & & & \\ B_{jn} & & & \end{array}\right]
\end{array} \qquad (j=1,\dots,n). \tag{3.3.b}$$

Here the symbol $\overset{j}{\smile}$ etc. designates the $(j+1)$th column etc. Let $\mathscr{I}$ denote the left $\mathscr{D}_X$-module $\{R(t,\partial/\partial t)\in M_{2n+1}(\mathscr{D}_X);\ R(t,\partial/\partial t)\,\mathbf{f}(x\mid t)=0\}$ and define $\mathscr{N}$ by $\mathscr{D}_X^{2n+1}/\mathscr{I}$. It is then clear that the following relations hold:

$$\frac{\partial}{\partial x_j}\mathbf{f}(x\mid t) = P_j\mathbf{f}(x\mid t) \qquad (j=1,\dots,n) \tag{3.4.a}$$

$$2\pi\sqrt{-1}\,x_k\mathbf{f}(x\mid t) = -P_{k+n}\mathbf{f}(x\mid t) \qquad (k=1,\dots,n). \tag{3.4.b}$$

Hence we obtain the following:

For $R\left(t,\dfrac{\partial}{\partial t}\right)$ in $\mathscr{I}$, $R\left(t,\dfrac{\partial}{\partial t}\right)P_j\left(t,\dfrac{\partial}{\partial t}\right)\mathbf{f}(x\mid t)$

$$= R\left(t,\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x_j}\mathbf{f}(x\mid t) = \frac{\partial}{\partial x_j}R\left(t,\frac{\partial}{\partial t}\right)\mathbf{f}(x\mid t)$$

$$= 0\ (j=1,\dots,n)\ \text{holds}. \tag{3.5.a}$$

For $R\left(t,\dfrac{\partial}{\partial t}\right)$ in $\mathscr{I}$, $R\left(t,\dfrac{\partial}{\partial t}\right)P_{k+n}\left(t,\dfrac{\partial}{\partial t}\right)\mathbf{f}(x\mid t)$

$$=R\left(t,\frac{\partial}{\partial t}\right)(-2\pi\sqrt{-1}\,x_k)\,\mathbf{f}(x\mid t)$$

$$=-2\pi\sqrt{-1}\,x_k R\left(t,\frac{\partial}{\partial t}\right)\mathbf{f}(x\mid t)=0\ (k=1,\ldots,n)\text{ holds.}$$

(3.5.b)

$$\begin{aligned}0&=[\partial/\partial x_j-P_j(t,\partial/\partial t),\partial/\partial x_{j'}-P_{j'}(t,\partial/\partial t)]\,\mathbf{f}(x\mid t)\\&=[P_j(t,\partial/\partial t),P_{j'}(t,\partial/\partial t)]\,\mathbf{f}(x\mid t)\qquad(1\leqslant j,j'\leqslant n).\end{aligned}\tag{3.5.c}$$

$$\begin{aligned}0&=[\partial/\partial x_j-P_j(t,\partial/\partial t),2\pi\sqrt{-1}\,x_k+P_{k+n}(t,\partial/\partial t)]\,\mathbf{f}(x\mid t)\\&=(2\pi\sqrt{-1}\,\delta_{jk}-[P_j(t,\partial/\partial t),P_{k+n}(t,\partial/\partial t)])\,\mathbf{f}(x\mid t)\qquad(1\leqslant j,k\leqslant n).\end{aligned}\tag{3.5.d}$$

$$\begin{aligned}0&=[2\pi\sqrt{-1}\,x_k+P_{k+n}(t,\partial/\partial t),2\pi\sqrt{-1}\,x_{k'}+P_{k'+n}(t,\partial/\partial t)]\,\mathbf{f}(x\mid t)\\&=[P_{k+n}(t,\partial/\partial t),P_{k'+n}(t,\partial/\partial t)]\,\mathbf{f}(x\mid t)\qquad(1\leqslant k,k'\leqslant n).\end{aligned}\tag{3.5.e}$$

These imply that $\mathscr{I}P_j\subset\mathscr{I}$ $(j=1,\ldots,2n)$ holds and that $(2\pi\sqrt{-1}\,\delta_{jk}-[P_j,P_{k+n}])$ etc. belongs to $\mathscr{I}$. Therefore conditions (2.1) and (2.3) are satisfied with the structure matrix

$$E=\begin{bmatrix}0 & -I_n\\ I_n & 0\end{bmatrix}$$

for the pair $P=\{P_j\}_{j=1,\ldots,2n}$ and $\mathscr{N}$. Hence it suffices to verify (2.2) to claim that P is a Jacobi structure. Since $Q=_{\text{def}}\sum_{j=1}^{2n}c_jP_j$ has the form

$$\begin{bmatrix}0 & c_1 & \cdots & c_{2n}\\ L_1 & & & \\ \vdots & & 0 & \\ L_{2n} & & & \end{bmatrix}$$

with L_j $(j=1,\ldots,2n)$ being a linear differential operator of order 1, Q^2 has the form

$$\begin{bmatrix}\sum_{j=1}^{2n}c_jL_j & & 0 & \\ & c_1L_1 & \cdots & c_{2n}L_1\\ 0 & \vdots & & \\ & c_1L_{2n} & \cdots & c_{2n}L_{2n}\end{bmatrix}.$$

This means ord $Q\leqslant 1/2$. Thus condition (2.2) is also verified.

Now let us define a left ideal $\mathscr{I}_0$ of $\mathscr{D}_X$ by

$$\{S(t, \partial/\partial t) \in \mathscr{D}_X; S(t, \partial/\partial t) f_0(x \mid t) = 0\} \tag{3.6}$$

and denote by V the characteristic variety of $\mathscr{D}_X/\mathscr{I}_0$. Define a subvariety Λ of T^*X, the cotangent bundle of X, by

$$\{(t, \tau) \in V; \sigma_1(A_{jj'})(t, \tau) = 0, \sigma_1(B_{kk'})(t, \tau) = 0 \ (1 \leqslant j, j', k, k' \leqslant n)\} \tag{3.7}$$

and assume

$$\Lambda \text{ is Lagrangian.} \tag{3.8}$$

Here $\sigma_1(A_{jj'})$ etc. denote the principal symbol of $A_{jj'}$ etc. Then we can verify that the solution space of the following Eqs. (3.9) is finite dimensional, and hence $\dim_{\mathbb{C}} J(P, c)$ $(c \in \mathbb{C}^{2n})$ is finite. (See Remark 2.16.)

$$\begin{aligned} &\frac{\partial}{\partial x_j} \mathbf{f}(x \mid t) = P_j \mathbf{f}(x \mid t) && (j = 1,\ldots, n), \\ &2\pi \sqrt{-1}\, x_k \mathbf{f}(x \mid t) = -P_{k+n} \mathbf{f}(x \mid t) && (k = 1,\ldots, n), \\ &R(t, \partial/\partial t)\, \mathbf{f}(x \mid t) = 0 && (R \in \mathscr{I}). \end{aligned} \tag{3.9}$$

In fact, as each solution $f(x \mid t)$ of (3.8) is uniquely determined by $\mathbf{f}(0 \mid t)$, it suffices to verify the finite dimensionality of the vector space spanned by $\mathbf{f}(0 \mid t)$. Since the $(j + 1)$th component $f_j(0 \mid t)$ of $\mathbf{f}(0 \mid t)$ is zero for $j = 1,\ldots, n$, it suffices to study $f_j(0 \mid t)$ for $j = 0$ and $j = n + 1,\ldots, 2n$. Let us now note that $S(t, \partial/\partial t)\, I_{2n+1}$ belongs to $\mathscr{I}$ if S belongs to $\mathscr{I}_0$. Hence we have

$$\begin{aligned} A_{kl} f_0(0 \mid t) &= 0 && (1 \leqslant k, l \leqslant n), \\ B_{k'l'} f_0(0 \mid t) &= 0 && (1 \leqslant k', l' \leqslant n), \\ S f_0(0 \mid t) &= 0 && (S \in \mathscr{I}_0). \end{aligned} \tag{3.10}$$

Then assumption (3.8) implies that the system that $f_0(0 \mid t)$ satisfies is holonomic.

Next let us study $f_j(0 \mid t)$ for $j = n + 1,\ldots, 2n$. It follows from (3.4.a) and (3.4.b) that we have

$$-4\pi^2 \frac{\partial}{\partial x_l} (x_k x_j \mathbf{f}(x \mid t)) = P_{j+n} P_{k+n} P_l \mathbf{f}(x \mid t) \qquad (1 \leqslant j, k, l \leqslant n). \tag{3.11}$$

Here we note

$$-4\pi^2 \frac{\partial}{\partial x_l}(x_k x_j \mathbf{f}(x \mid t))$$

$$= -4\pi^2 x_j x_k \frac{\partial}{\partial x_l} \mathbf{f}(x \mid t) + x_j \delta_{kl} \mathbf{f}(x \mid t) + x_k \delta_{lj} \mathbf{f}(x \mid t) \qquad (3.12)$$

and

$$P_{j+n} P_{k+n} P_l = -4\pi^2 \begin{array}{c} \begin{array}{cccc} & & \overbrace{}^{k} & \end{array} \\ \begin{bmatrix} A_{kj} & 0 & 0 & \\ & & A_{j1} & 0 \\ 0 & 0 & \vdots & \\ & & B_{jn} & \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cccc} & & \overbrace{}^{n+l} & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ B_{l1} & & & \\ \vdots & & 0 & \\ C_{nl} & & & \end{bmatrix} \end{array}. \qquad (3.13)$$

In view of (3.12) and (3.13), the comparison of the first component of (3.11) entails

$$A_{kj} f_{n+l}(0 \mid t) = 0 \qquad (1 \leqslant j, k, l \leqslant n). \qquad (3.14)$$

Similarly, by calculating $(\partial/\partial x_l)(x_k(\partial/\partial x_j)\,\mathbf{f}(x \mid t))$, we obtain

$$B_{kj} f_{n+l}(0 \mid t) = \delta_{kl} f_{n+j}(0 \mid t). \qquad (3.15)$$

Since $S f_{n+l}(0 \mid t) = 0$ holds for any S in $\mathscr{I}_0$, (3.14) and (3.15) imply that $f_{n+l}(0 \mid t)$ satisfies a holonomic system. Thus we have shown the required finiteness property of $\{\mathbf{f}(0 \mid t)\}$.

So far we have considered the problem starting from (3.2.a) ~ (3.2.c). We can, however, construct a Jacobi structure by the same method in a more general situation, that is, starting from the following (3.16.a) ~ (3.16.c) instead of (3.2.a) ~ (3.2.c):

$$x_{j_1} \cdots x_{j_m} f_0(x \mid t) = A_{j_1 \ldots j_m}(t, \partial/\partial t) f_0(x \mid t) \qquad (1 \leqslant j_1, \ldots, j_m \leqslant n), \qquad (3.16.a)$$

$$x_j \frac{\partial}{\partial x_k} f_0(x \mid t) = B_{jk}(t, \partial/\partial t) f_0(x \mid t), \qquad (3.16.b)$$

$$\frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}} f_0(x \mid t) = C_{j_1 \ldots j_m}(t, \partial/\partial t) f_0(x \mid t) \qquad (1 \leqslant j_1, \ldots, j_m \leqslant n), \qquad (3.16.c)$$

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where $\operatorname{ord} B_{jk} = 1$, $\operatorname{ord} A_{j_1 \cdots j_m} = \alpha$ and $\operatorname{ord} C_{j_1 \cdots j_m} = \beta$ $(1 \leqslant j, \ k \leqslant n,$ $1 \leqslant j_1, \ldots, j_m \leqslant n)$ with $\alpha + \beta = m$ and $\alpha, \beta < m$.

Since the argument is the same as before, we leave the detailed calculation to the reader.

We end this section by giving simple examples of $f_0(x \mid t)$ which satisfy condition (3.2).

EXAMPLE 1. Let t denote a symmetric matrix $(t_{jk})_{1 \leqslant j,k \leqslant n}$ and let $f_0(x \mid t)$ be given by $\exp(\pi \sqrt{-1}(\sum_{1 \leqslant j,k \leqslant n} t_{jk} x_j x_k))$. It is then clear that $f_0(x \mid t)$ satisfies conditions (3.1) and (3.2). In this case, one can verify that the first component of the resulting Jacobi function with $c_j = 1$ $(j = 1, \ldots, 2n)$ is a constant multiple of the zero-value of the Riemann theta function $\sum_{\nu \in \mathbb{Z}^n} \exp(\pi \sqrt{-1}(\sum_{j,k} t_{jk} \nu_j \nu_k))$.

EXAMPLE 2 (cf. Weil [7]). Let A, B and C be $n \times n$ constant matrices. Suppose that A is invertible and that $B = (b_{ij})_{1 \leqslant i,j \leqslant n}$ and $C = (c_{ij})_{1 \leqslant i,j \leqslant n}$ are symmetric. Denote $\sum_{i,j} b_{ij} y_i y_j$ and $\sum_{i,j} c_{ij} \xi_i \xi_j$ by $b(y)$ and $c(\xi)$, respectively. Let $\Phi(x, y, \xi; A, B, C)$ denote

$$\exp(-2\pi \sqrt{-1} \langle Ax - y, \xi \rangle - \pi \sqrt{-1}\, c(\xi) + \pi \sqrt{-1}\, b(y)).$$

Let $\varphi(y)$ be a tempered hyperfunction defined on $\mathbb{R}^n$ and define $\mathscr{W}(\varphi)(x \mid A, B, C)$ by

$$\iint \Phi(x, y, \xi; A, B, C)\, \varphi(y)\, dy\, d\xi. \tag{3.17}$$

Then, by choosing A, B and C as t-variables, one can verify that $\mathscr{W}(\varphi)(x \mid A, B, C)$ satisfies the condition (3.2) if $\operatorname{Im} B$ and $-\operatorname{Im} C$ are sufficiently large.

More detailed discussion on this example will be given elsewhere.

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